

# TOWARD BEST ISOPERIMETRIC CONSTANTS FOR ( $H^1, BMO$ )-NORMAL CONFORMAL METRICS ON $\mathbb{R}^n$ , $n \geq 3$ <sup>†</sup>

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ABSTRACT. The aim of this article is: (a) To establish the existence of the best isoperimetric constants for the  $(H^1, BMO)$ -normal conformal metrics  $e^{2u}|dx|^2$  on  $\mathbb{R}^n$ ,  $n \geq 3$ , i.e., the conformal metrics with the  $Q$ -curvature orientated conditions

$$(-\Delta)^{n/2}u \in H^1(\mathbb{R}^n) \quad \& \quad u(x) = \text{const.} + \frac{\int_{\mathbb{R}^n} (\log \frac{|\cdot|}{|x-\cdot|}) (-\Delta)^{n/2}u(\cdot) d\mathcal{H}^n(\cdot)}{2^{n-1}\pi^{n/2}\Gamma(n/2)};$$

(b) To prove that  $(n\omega_n^{\frac{1}{n}})^{\frac{n}{n-1}}$  is the optimal upper bound of the best isoperimetric constants for the complete  $(H^1, BMO)$ -normal conformal metrics with nonnegative scalar curvature; (c) To find the optimal upper bound of the best isoperimetric constants via the quotients of two power integrals of Green's functions for the  $n$ -Laplacian operators  $-\text{div}(|\nabla u|^{n-2}\nabla u)$ .

## 1. INTRODUCTION

The original motivation of this paper goes back to one of the geometric  $Q$ -curvature problems posed on Lawrence J. Peterson's edited article – Future Directions of Research in Geometry: A Summary of the Panel Discussion at the 2007 Midwest Geometry Conference (cf. [28]).

**Alice Chang's Question:** *A very general question is to ask “What is the geometric content of  $Q$ -curvature?” For example, we know that one can associate the scalar curvature with the conformally invariant constant called the “Yamabe constant”. When this constant is positive, it describes the best constant (in a conformally invariant sense) of the Sobolev embedding of  $W^{1,2}$  into  $L^{2n/(n-2)}$  space; this in itself can be viewed as a  $W^{1,2}$  version of the isoperimetric inequality. It would be interesting to know if  $Q$ -curvature, or the conformally invariant quantity  $\int Q$  associated with it, satisfies some similar inequalities with geometric content.*

To find out a way to attack this question let us choose a conformally flat manifold  $(\mathbb{R}^n, g)$  as the acting model – the  $2 \leq n$ -dimensional Euclidean space  $\mathbb{R}^n$  equipped with the conformal metric  $g = e^{2u}g_0$ , where  $u$  is a real-valued smooth function on  $\mathbb{R}^n$ , i.e.,  $u \in C^\infty(\mathbb{R}^n)$ , and  $g_0 = |dx|^2 = \sum_{k=1}^n dx_k^2$  is the standard Euclidean metric on  $\mathbb{R}^n$ . For the convenience of statement let us also agree to several more basic conventions. The symbols  $\Delta$  and  $\nabla$  denote the Laplace operator  $\sum_{k=1}^n \partial^2/\partial x_k^2$  and the gradient vector  $(\partial/\partial x_1, \dots, \partial/\partial x_n)$  over  $\mathbb{R}^n$ . The volume and surface area elements of the metric  $g$  are determined via

$$dv_{g,n} = e^{nu} d\mathcal{H}^n \quad \text{and} \quad ds_{g,n} = e^{(n-1)u} d\mathcal{H}^{n-1}$$

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where  $\mathcal{H}^k$  stands for the  $k$ -dimensional Hausdorff measure on  $\mathbb{R}^n$ . Thus, the volume and surface area of the open ball  $B_r(x)$  and its boundary  $\partial B_r(x)$  with radius  $r > 0$  and center  $x \in \mathbb{R}^n$  take the following values:

$$v_{g,n}(B_r(x)) = \int_{B_r(x)} e^{nu} d\mathcal{H}^n \quad \text{and} \quad s_{g,n}(\partial B_r(x)) = \int_{\partial B_r(x)} e^{(n-1)u} d\mathcal{H}^{n-1}.$$

At the same time, on the conformally flat manifold  $(\mathbb{R}^n, g)$  there are two types of curvature – one is the Ricci's scalar curvature

$$S_{g,n} = -2(n-1)e^{-2u} \left( \Delta u + \frac{n-2}{2} |\nabla u|^2 \right);$$

and the other is the Paneitz's Q-curvature which, according as [11] and [26], is given by

$$Q_{g,n} = e^{-nu} (-\Delta)^{n/2} u.$$

Here and hereafter, for  $\alpha \in \mathbb{R}$  the operator  $(-\Delta)^{\alpha/2}$  is initially defined via the Fourier transform

$$\widehat{(-\Delta)^{\alpha/2} f}(x) = (2\pi|x|)^{\alpha} \hat{f}(x) = (2\pi|x|)^{\alpha} \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} f(y) d\mathcal{H}^n(y),$$

where  $f$  is of the Schwartz class, denoted  $f \in \mathcal{S}(\mathbb{R}^n)$ , that is,

$$f \in C^\infty(\mathbb{R}^n) \quad \text{and} \quad \sup_{x=(x_1, \dots, x_n) \in \mathbb{R}^n} (1+|x|)^N \left| \frac{\partial^{k_1+\dots+k_n} f}{\partial^{k_1} x_1 \dots \partial^{k_n} x_n}(x) \right| < \infty$$

for all multi-indices  $(k_1, \dots, k_n)$  and natural numbers  $N$ . Of course, the domain of  $(-\Delta)^{\alpha/2}$  can be extended to  $C^\infty(\mathbb{R}^n)$  via the duality pairing:

$$\langle (-\Delta)^{\alpha/2} f, h \rangle = \langle f, (-\Delta)^{\alpha/2} h \rangle \quad \text{where} \quad f \in C^\infty(\mathbb{R}^n) \quad \text{and} \quad h \in \mathcal{S}(\mathbb{R}^n).$$

In addition to the operators  $S_{g,n}$  and  $Q_{g,n}$ , there is the third operator related to the Laplacian, that is, the  $n$ -Laplacian

$$\Delta_n u = -\operatorname{div}(|\nabla u|^{n-2} \nabla u).$$

Associated with this operator is the  $n$ -Green function  $G_\Omega(\cdot, \cdot)$  of a domain  $\Omega \subset \mathbb{R}^n$  with the boundary  $\partial\Omega \neq \emptyset$ , that is, the weak solution to the Dirichlet problem:

$$\begin{cases} \Delta_n G_{\Omega,n}(x, y) = \delta_y(x) & , \quad x \in \Omega \\ G_{\Omega,n}(x, y) = 0 & , \quad x \in \partial\Omega. \end{cases}$$

Here  $\delta_y(x)$  is the Dirac measure. Of course, such a weak solution does not always exist. Consequently, when a domain is bounded and has the  $n$ -Green's function, the domain is said to be bounded regular.

Since the scalar curvature  $S_{g,2}/2$  and the Q-curvature  $Q_{g,2}$  coincide with the classical Gaussian curvature  $K$ :

$$\frac{S_{g,2}}{2} = Q_{g,2} = e^{-2u} (-\Delta) u = e^{-2u} \Delta_2 u = K$$

which completely characterizes the curvature of the two-dimensional conformally flat manifold  $(\mathbb{R}^2, g)$ , Chang's question leads us to recall an easily-verified consequence of Li-Tam's isoperimetric inequality (cf. [23, Theorems 5.1-5.2 & Corollary 5.3]), Finn's isoperimetric deficit formula [12] and Huber's isoperimetric inequality [17, Theorem 3]:

**Two-dimensional Theorem:** For  $u \in C^\infty(\mathbb{R}^2)$  suppose  $g = e^{2u}g_0$  is a conformal metric on  $\mathbb{R}^2$ . Let

$$(1.1) \quad \int_{\mathbb{R}^2} |Q_{g,2}| dv_{g,2} < \infty \quad \text{and} \quad \int_{\mathbb{R}^2} Q_{g,2} dv_{g,2} < 2\pi.$$

Then

(i)

$$(1.2) \quad \kappa_{g,2} = \inf_{\Omega} \frac{(s_{g,2}(\partial\Omega))^2}{v_{g,2}(\Omega)} = \inf_f \frac{\left( \int_{\mathbb{R}^2} |\nabla f| dv_{g,2} \right)^2}{\int_{\mathbb{R}^2} |f|^2 dv_{g,2}}$$

is a positive number depending only on  $(\mathbb{R}^2, g)$ , where the left-hand infimum is taken over all pre-compact domains  $\Omega \subseteq \mathbb{R}^2$  with  $C^1$ -boundary  $\partial\Omega$ , and the right-hand infimum ranges over all  $C^1$ -functions  $f$  with compact support in  $\mathbb{R}^2$ .

(ii)

$$(1.3) \quad \kappa_{g,2} = 2 \left( 2\pi - \int_{\mathbb{R}^2} Q_{g,2} dv_{g,2} \right)$$

holds for  $Q_{g,2} \geq 0$ , where  $\kappa_{g,2} = 4\pi$  if and only if  $g = g_0$ .

Clearly, an appropriate higher-dimensional analogue of the previously-quoted two-dimensional theorem (including condition (1.1) and assertions (i)-(ii)) would suggest a solution to Chang's question for the Euclidean manifold  $(\mathbb{R}^n, g)$ . For future use, the symbol  $H^1(\mathbb{R}^n)$  (cf. [14, Theorem 6.7.4]) denotes the Hardy space of all real-valued functions  $f$  on  $\mathbb{R}^n$  that satisfy

$$\|f\|_{H^1} = \int_{\mathbb{R}^n} |f| d\mathcal{H}^n + \sum_{j=1}^n \int_{\mathbb{R}^n} |R_j(f)| d\mathcal{H}^n < \infty,$$

where the Riesz transforms

$$R_j(f)(x) = \lim_{\epsilon \rightarrow 0} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_{|y| \geq \epsilon} y_j |y|^{-n-1} f(x-y) d\mathcal{H}^n(y), \quad j = 1, \dots, n$$

are well-determined for  $f \in L^1(\mathbb{R}^n)$  and the classical gamma function  $\Gamma(\cdot)$ .

In addition, the best isoperimetric constant for a given conformal metric  $g$  on  $\mathbb{R}^n$  is defined by

$$(1.4) \quad \kappa_{g,n} = \inf_{\Omega \in BDC(\mathbb{R}^n)} \frac{(s_{g,n}(\partial\Omega))^{\frac{n}{n-1}}}{v_{g,n}(\Omega)},$$

where  $BDC(\mathbb{R}^n)$  represents the class of all bounded domains  $\Omega \subset \mathbb{R}^n$  with  $C^1$ -smooth boundary  $\partial\Omega$ .

According to Chang's question as well as (1.2), our focus should be on deciding when the sharp constant in (1.4) is positive. Below is the outcome.

**Theorem 1.1.** For  $u \in C^\infty(\mathbb{R}^n)$  suppose  $g = e^{2u}g_0$  is a conformal metric on  $\mathbb{R}^n$ ,  $n \geq 3$ . If  $g$  is  $(H^1, BMO)$ -normal, namely, if

$$(1.5) \quad (-\Delta)^{n/2} u \in H^1(\mathbb{R}^n)$$

and there is a constant  $c$  such that

$$(1.6) \quad u(x) = c + \frac{\int_{\mathbb{R}^n} \left( \log \frac{|y|}{|x-y|} \right) (-\Delta)^{n/2} u(y) d\mathcal{H}^n(y)}{2^{n-1} \pi^{n/2} \Gamma(n/2)} \quad \text{for } x \in \mathbb{R}^n,$$

then

$$(1.7) \quad 0 < \kappa_{g,n} = \inf_{f \in C_0^1(\mathbb{R}^n)} \frac{\left( \int_{\mathbb{R}^n} |\nabla f| dv_{g,n} \right)^{\frac{n}{n-1}}}{\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} dv_{g,n}} < \infty,$$

where the infimum ranges over  $f \in C^1(\mathbb{R}^n)$  with compact support in  $\mathbb{R}^n$ .

Perhaps it is worth pointing out that the notion of  $(H^1, BMO)$ -normal is naturally inspired by both (1.5) which amounts to the following Q-curvature constraint:

$$\int_{\mathbb{R}^n} \left( |Q_{g,n}| + e^{-nu} |\nabla(e^{(n-1)u} Q_{g,n-1})| \right) dv_{g,n} < \infty$$

and the famous C. Fefferman's duality  $[H^1(\mathbb{R}^n)]^* = BMO(\mathbb{R}^n)$ , John-Nirenberg's space of functions with bounded mean oscillation in  $\mathbb{R}^n$  (cf. [10]), which contains the function  $\log |\cdot|/|x - \cdot|$  for any fixed  $x \in \mathbb{R}^n$ . Here it is also worth mentioning that the conditions

$$\int_{\mathbb{R}^n} |Q_{g,n}| dv_{g,n} < \infty \quad \text{and} \quad (1.6)$$

produce the definition for a conformal metric to be (classical) normal – see also [12] for  $n = 2$ ; [5, Definition 3.1] & [6, Definition 1.7] for  $n = 4$ ; [9] & [4] for even integer  $n \geq 4$ ; [27] & [35] for any integer  $n \geq 3$ . Obviously, the  $(H^1, BMO)$ -normal is stronger than the normal. From [18], [27] and [35] it turns out that any conformal metric  $g$  on  $\mathbb{R}^n$  with  $n \geq 2$  satisfying

$$(1.8) \quad \int_{\mathbb{R}^n} |Q_{g,n}| dv_{g,n} < \infty \quad \text{and} \quad \lim_{|x| \rightarrow \infty} \inf_{|y| > |x|} S_{g,n}(y) \geq 0$$

is normal.

As a first application of Theorem 1.1, we obtain the following result (cf. (1.3)) which seems most closely tied to Chang's question above.

**Theorem 1.2.** *For  $u \in C^\infty(\mathbb{R}^n)$  suppose  $g = e^{2u} g_0$  is a complete conformal metric on  $\mathbb{R}^n$ ,  $n \geq 3$ , with*

$$(1.9) \quad (-\Delta)^{n/2} u \in H^1(\mathbb{R}^n) \quad \text{and} \quad S_{g,n} \geq 0.$$

Then

$$(1.10) \quad 0 < \frac{\kappa_{g,n}}{(n\omega_n^{\frac{1}{n}})^{\frac{n}{n-1}}} \leq 1 - \frac{\int_{\mathbb{R}^n} Q_{g,n} dv_{g,n}}{2^{n-1} \Gamma(n/2) \pi^{n/2}} = 1,$$

where

$$\omega_n = \mathcal{H}^n(B_1(0)) = 2\pi^{n/2} (n\Gamma(n/2))^{-1}$$

is the  $n$ -dimensional Hausdorff measure of the unit ball  $B_1(0)$  of  $\mathbb{R}^n$ . Moreover, the relation “ $\leq$ ” in (1.10) becomes the relation “ $=$ ” if and only if  $g = g_0$ .

As a second application of Theorem 1.1, we gain the optimal upper bound of  $\kappa_{g,n}$  through a comparison between two integrals of the Green function associated with the  $n$ -Laplacian operator.

**Theorem 1.3.** *For  $u \in C^\infty(\mathbb{R}^n)$  let  $g = e^{2u} g_0$  be an  $(H^1, BMO)$ -normal conformal metric on  $\mathbb{R}^n$ ,  $n \geq 3$ . Suppose  $BRD(\mathbb{R}^n)$  stands for the class of all bounded regular domains  $\Omega \subset \mathbb{R}^n$ . Then*

(i)

$$(1.11) \quad 0 < \frac{\kappa_{g,n}^{-q} \Gamma(q+1)}{\kappa_{g,n}^{-p} \Gamma(p+1)} \leq \inf_{x \in \Omega \in BRD(\mathbb{R}^n)} \frac{\int_{\Omega} (G_{\Omega,n}(x, y))^q dv_{g,n}(y)}{\int_{\Omega} (G_{\Omega,n}(x, y))^p dv_{g,n}(y)} < \infty$$

holds for  $0 \leq q < p < \infty$ . Moreover, the equality in (1.11) is valid for  $g = g_0$ .

(ii)

$$(1.12) \quad 0 < \frac{\kappa_{g,n}^{p+1}}{\Gamma(p+1)} \leq \inf_{x \in \Omega \in BRD(\mathbb{R}^n)} \frac{(s_{g,n}(\partial\Omega))^{\frac{n}{n-1}}}{\int_{\Omega} (G_{\Omega,n}(x, y))^p dv_{g,n}(y)} < \infty$$

holds for  $0 \leq p < \infty$ . Moreover, the equality in (1.12) holds for  $g = g_0$ .

The proofs of Theorems 1.1-1.2-1.3 are provided in the second, third and fourth sections respectively. Our techniques and methods are of strong harmonic analysis flavor and developed partially on the basis of the following works: [2], [3], [5], [8], [9], [26], [27], and [32]. Here we would like to thank P. Li for sending us the motive paper [23], A. Chang and G. Zhang for reading the original version of this article, and the referee for giving us helpful suggestions.

## 2. PROOF OF THEOREM 1.1

To prove Theorem 1.1, we begin with the concept of David-Semmes' strong  $A_{\infty}$ -weight (cf. [8]).

### Definition 2.1.

(i) A function  $w : \mathbb{R}^n \rightarrow [0, \infty)$  is called an  $A_{\infty}$ -weight provided there are constants  $\epsilon > 0$  and  $C \geq 1$  such that

$$\left( (\mathcal{H}^n(B))^{-1} \int_B w^{1+\epsilon} d\mathcal{H}^n \right)^{\frac{1}{1+\epsilon}} \leq C (\mathcal{H}^n(B))^{-1} \int_B w d\mathcal{H}^n$$

holds for all Euclidean balls  $B \subset \mathbb{R}^n$ .

(ii) A nonnegative Borel measure  $\mu$  on  $\mathbb{R}^n$  is called a doubling measure provided there is a constant  $C \geq 1$  such that  $\mu(2B) \leq C\mu(B)$  holds for every Euclidean ball  $B = B_r(x) \subset \mathbb{R}^n$  and its doubling ball  $2B = B_{2r}(x)$ .

(iii) A doubling measure  $\mu$  on  $\mathbb{R}^n$  is called a metric doubling measure provided there are a metric  $d_{\mu}(\cdot, \cdot)$  on  $\mathbb{R}^n$  and a constant  $C \geq 1$  such that

$$C^{-1} d_{\mu}(x, y) \leq \mu(B_{|x-y|}(x) \cup B_{|y-x|}(y)) \leq C d_{\mu}(x, y) \quad \text{for } x, y \in \mathbb{R}^n.$$

In this case, there exists an  $A_{\infty}$ -weight  $w$  on  $\mathbb{R}^n$  such that  $d\mu = w d\mathcal{H}^n$  – such a weight is said to be a strong  $A_{\infty}$ -weight.

It is well-known that if  $w$  is an  $A_{\infty}$ -weight then  $u = \log w \in BMO(\mathbb{R}^n)$ :

$$\|u\|_{BMO} = \sup_B (\mathcal{H}^n(B))^{-1} \int_B \left| u - (\mathcal{H}^n(B))^{-1} \int_B u d\mathcal{H}^n \right| d\mathcal{H}^n < \infty,$$

where the supremum is taken over all Euclidean balls  $B \subset \mathbb{R}^n$ , and conversely, if  $u \in BMO(\mathbb{R}^n)$  then there is a constant  $c > 0$  depending on  $n$  and  $\|u\|_{BMO}$  such that  $w = e^{cu}$  is an  $A_{\infty}$ -weight. Moreover, a typical example of the strong  $A_{\infty}$ -weight is the Jacobian determinant  $J_f$  of a quasiconformal mapping  $f$  of  $\mathbb{R}^n$  onto

itself in that if  $d_\mu(x, y) = |f(x) - f(y)|$  then a change of variables plus a distortion structure of quasiconformal mappings (cf. [16, p.380]) gives

$$d_\mu(x, y) \approx \left( \mathcal{H}^n \left( f(B_{|x-y|}(x) \cup B_{|y-x|}(y)) \right) \right)^{\frac{1}{n}} \approx \left( \int_{B_{|x-y|}(x) \cup B_{|y-x|}(y)} J_f d\mathcal{H}^n \right)^{\frac{1}{n}}.$$

Here and henceafter,  $X \approx Y$  means  $C^{-1}Y \leq X \leq CY$  for a constant  $C \geq 1$  independent of  $X$  and  $Y$ , and moreover the symbol  $X \lesssim Y$  stands for  $X \leq CY$ .

The lemma below is a straightforward consequence of David-Semmes' [8, (2.4)].

**Lemma 2.2.** *If  $w$  is a strong  $A_\infty$ -weight, then there is a constant  $C > 0$  such that the isoperimetric inequality*

$$\int_{\Omega} w d\mathcal{H}^n \leq C \left( \int_{\partial\Omega} w^{\frac{n-1}{n}} d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}}$$

holds for every bounded open set  $\Omega \subset \mathbb{R}^n$ .

From Bonk-Heinonen-Saksman's [3, Theorem 3.1 & Remark 3.26] we can readily obtain the following result.

**Lemma 2.3.** *Given  $\alpha \in (0, n)$  and  $x \in \mathbb{R}^n$  let*

$$u(x) = (I_\alpha f)(x) = \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} d\mathcal{H}^n(y)$$

converge for some function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$  with

$$\|f\|_{L^{n/\alpha}} = \left( \int_{\mathbb{R}^n} |f|^{n/\alpha} d\mathcal{H}^n \right)^{\alpha/n} < \infty.$$

Then  $w = e^{nu}$  is a strong  $A_\infty$ -weight.

The forthcoming technical result is also useful.

**Lemma 2.4.** *Let  $0 < \lambda < n$ . Then*

$$\sup_{(r, x, y) \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n} \frac{r^\lambda}{\mathcal{H}^n(B_r(x))} \int_{B_r(x)} |z-y|^{-\lambda} d\mathcal{H}^n(z) < \infty.$$

*Proof.* Using a dyadic portion of  $B_r(y)$  we estimate

$$\begin{aligned} & \left( \mathcal{H}^n(B_r(x)) \right)^{-1} \int_{B_r(x)} \frac{d\mathcal{H}^n(z)}{|z-y|^\lambda} \\ & \approx r^{-n} \left( \int_{B_r(x) \cap (\mathbb{R}^n \setminus B_r(y))} \frac{d\mathcal{H}^n(z)}{|z-y|^\lambda} + \int_{B_r(x) \cap B_r(y)} \frac{d\mathcal{H}^n(z)}{|z-y|^\lambda} \right) \\ & \lesssim r^{-(n+\lambda)} \mathcal{H}^n(B_r(x) \cap (\mathbb{R}^n \setminus B_r(y))) \\ & \quad + r^{-n} \sum_{k=0}^{\infty} \int_{B_r(x) \cap (B_{2^{-k}r}(y) \setminus B_{2^{-k-1}r}(y))} \frac{d\mathcal{H}^n(z)}{|z-y|^\lambda} \\ & \lesssim r^{-\lambda} \left( 1 + r^{-n} \sum_{k=0}^{\infty} 2^{k\lambda} \mathcal{H}^n(B_r(x) \cap (B_{2^{-k}r}(y) \setminus B_{2^{-k-1}r}(y))) \right) \\ & \lesssim r^{-\lambda} \left( 1 + \sum_{k=0}^{\infty} 2^{-k(n-\lambda)} \right), \end{aligned}$$

whence getting the desired finiteness.  $\square$

**Proof of Theorem 1.1.** We first prove  $0 < \kappa_{g,n} < \infty$ . Using  $(-\Delta)^{n/2}u \in H^1(\mathbb{R}^n)$ , the celebrated Stein-Weiss-Krantz's boundedness of  $I_\alpha : H^1(\mathbb{R}^n) \rightarrow L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$  (cf. [31] and [21]), and  $(-\Delta)^{-\frac{1}{2}} = I_1$ , we gain

$$\begin{aligned}
 (2.1) \quad & \left( \int_{\mathbb{R}^n} |(-\Delta)^{\frac{n-1}{2}} u|^{\frac{n}{n-1}} d\mathcal{H}^n \right)^{\frac{n-1}{n}} \\
 &= \left( \int_{\mathbb{R}^n} |I_1(-\Delta)^{\frac{n}{2}} u|^{\frac{n}{n-1}} d\mathcal{H}^n \right)^{\frac{n-1}{n}} \\
 &\lesssim \|(-\Delta)^{n/2} u\|_{H^1}
 \end{aligned}$$

Note also that for  $n \geq 3$  and  $x \neq y$  (cf. [20, p.128, (2.10.1) & (2.10.8)] and [24, p.132, (3)]),

$$\begin{aligned}
 & (-\Delta)^{\frac{1}{2}} \log |x - y|^{-1} \\
 &= (-\Delta)^{-\frac{1}{2}} (-\Delta) \log |x - y| \\
 &= (n-2)I_1(|x - \cdot|^{-2})(y) \\
 &= \frac{(n-2)\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n}{2}}\Gamma(\frac{1}{2})} \int_{\mathbb{R}^n} |x - z|^{-2} |z - y|^{1-n} d\mathcal{H}^n(z) \\
 &= \left( \frac{\pi^{\frac{1}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \right) |x - y|^{-1}.
 \end{aligned}$$

So if

$$u_1(x) = I_{n-1}((-\Delta)^{\frac{n-1}{2}} u)(x) \quad \text{for } x \in \mathbb{R}^n,$$

then

$$\begin{aligned}
 & (-\Delta)^{\frac{n-1}{2}} u_1(x) \\
 &= \frac{\int_{\mathbb{R}^n} ((-\Delta)^{\frac{n-1}{2}} |x - y|^{-1}) (-\Delta)^{\frac{n-1}{2}} u(y) d\mathcal{H}^n(y)}{2^{n-1} \pi^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} \\
 &= \frac{\int_{\mathbb{R}^n} ((-\Delta)^{\frac{n-1}{2}} (-\Delta)^{\frac{1}{2}} \log |x - y|^{-1}) (-\Delta)^{\frac{n-1}{2}} u(y) d\mathcal{H}^n(y)}{2^{n-1} \pi^{\frac{n}{2}} \Gamma(\frac{n}{2})} \\
 &= \int_{\mathbb{R}^n} \delta_x(y) (-\Delta)^{\frac{n-1}{2}} u(y) d\mathcal{H}^n(y) \\
 &= (-\Delta)^{\frac{n-1}{2}} u(x)
 \end{aligned}$$

Here we have used the formula (cf. [27, Proposition 2.1 (iv)]) that

$$(-\Delta)^{n/2} (-\log |x - y|) = 2^{n-1} \Gamma(n/2) \pi^{n/2} \delta_x(y)$$

holds in the sense of distribution. Consequently,  $(-\Delta)^{\frac{n-1}{2}}(u - u_1) = 0$ . In other words,

$$0 = (2\pi|x|)^{n-1} (\widehat{u - u_1})(x), \quad x \in \mathbb{R}^n.$$

Since  $n \geq 3$ , this last equation forces  $(-\Delta)(u - u_1) = 0$ , namely,  $u - u_1$  is a harmonic function on  $\mathbb{R}^n$  and so is each coordinate of the vector  $\nabla(u - u_1)$ .

A combined application of (1.6), the mean-value property of  $\partial(u - u_1)(y)/\partial y_j$ , Fubini's theorem and Lemma 2.4 derives that for any  $r > 0$  and  $x \in \mathbb{R}^n$ ,

$$\begin{aligned}
& \left| \frac{\partial(u - u_1)}{\partial y_j}(x) \right| \\
&= \left| \left( \mathcal{H}^n(B_r(x)) \right)^{-1} \int_{B_r(x)} \frac{\partial(u - u_1)}{\partial y_j}(y) d\mathcal{H}^n(y) \right| \\
&\lesssim \int_{\mathbb{R}^n} \left( r^{-n} \int_{B_r(x)} \left| \frac{\partial}{\partial y_j} \log \frac{|z|}{|z - y|} \right| d\mathcal{H}^n(y) \right) |(-\Delta)^{n/2} u(z)| d\mathcal{H}^n(z) \\
&\quad + r^{-n} \int_{B_r(x)} \left| \int_{\mathbb{R}^n} \frac{\partial}{\partial y_j} \left( |z|^{-1} (-\Delta)^{\frac{n-1}{2}} u(y - z) \right) d\mathcal{H}^n(z) \right| d\mathcal{H}^n(y) \\
&\lesssim \int_{\mathbb{R}^n} \left( r^{-n} \int_{B_r(x)} |z - y|^{-1} d\mathcal{H}^n(y) \right) |(-\Delta)^{n/2} u(z)| d\mathcal{H}^n(z) \\
&\quad + \int_{\mathbb{R}^n} \left( r^{-n} \int_{B_r(x)} |z - y|^{-1} d\mathcal{H}^n(y) \right) |\nabla((-\Delta)^{(n-1)/2} u)(z)| d\mathcal{H}^n(z) \\
&\lesssim r^{-1} \left( \|(-\Delta)^{n/2} u\|_{L^1} + \|\nabla((-\Delta)^{(n-1)/2} u)\|_{L^1} \right) \\
&\lesssim r^{-1} \|(-\Delta)^{n/2} u\|_{H^1},
\end{aligned}$$

where we have also used the following formula (cf. [25, p.58, (1.94)]):

$$-R_j(f)(x) = \frac{\partial}{\partial x_j}(I_1 f)(x) = \frac{\partial}{\partial x_j}((-\Delta)^{-1/2} f)(x), \quad j = 1, 2, \dots, n.$$

Letting  $r \rightarrow \infty$  we obtain that  $\nabla(u - u_1)$  is the zero vector, whence finding that  $u - u_1$  is a constant  $c$ . Now we get by Lemma 2.3, (2.1) and the definition of  $u_1$  that  $w = e^{nu} = e^{nc} e^{nu_1}$  is a strong  $A_\infty$ -weight. This, together with Lemma 2.2, deduces that for any  $\Omega \in BDC(\mathbb{R}^n)$ ,

$$\int_{\Omega} e^{nu} d\mathcal{H}^n \leq C \left( \int_{\partial\Omega} e^{(n-1)u} d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}}$$

where  $C > 0$  is a constant independent of  $\Omega$ . Thus  $\kappa_{g,n}$  is a finite positive number.

Next, we prove

$$(2.2) \quad \kappa_{g,n} = \inf_{f \in C_0^1(\mathbb{R}^n)} \frac{\left( \int_{\mathbb{R}^n} |\nabla f| dv_{g,n} \right)^{\frac{n}{n-1}}}{\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} dv_{g,n}}.$$

In spite of being well-known, such an argument is included here for the completeness of the paper. For  $t \geq 0$  and  $f \in C_0^1(\mathbb{R}^n)$ , let

$$\Omega(t; f) = \{x \in \mathbb{R}^n : |f(x)| \geq t\},$$

then

$$\partial\Omega(t; f) = \{x \in \mathbb{R}^n : |f(x)| = t\}.$$

Thus, using the layer cake representation, the monotonicity of  $s_{g,n}(\partial\Omega(t; f))$  with respect to  $t \geq 0$  and the co-area formula for  $\nabla f$  (cf. [7, Theorem VIII.3.3]) we



obtain

$$\begin{aligned}
& \kappa_{g,n} \int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} dv_{g,n} \\
&= \kappa_{g,n} \int_0^\infty v_{g,n}(\Omega(t; f)) dt^{\frac{n}{n-1}} \\
&\leq \int_0^\infty \left( s_{g,n}(\partial\Omega(t; f)) \right)^{\frac{n}{n-1}} dt^{\frac{n}{n-1}} \\
&= \left( \frac{n}{n-1} \right) \int_0^\infty t^{\frac{1}{n-1}} \left( s_{g,n}(\partial\Omega(t; f)) \right)^{\frac{n}{n-1}} dt \\
&\leq \int_0^\infty \frac{d}{dt} \left( \left( \int_0^t s_{g,n}(\partial\Omega(r; f)) dr \right)^{\frac{n}{n-1}} \right) dt \\
&= \left( \int_0^\infty s_{g,n}(\partial\Omega(t; f)) dt \right)^{\frac{n}{n-1}} \\
&= \left( \int_{\mathbb{R}^n} |\nabla f| dv_{g,n} \right)^{\frac{n}{n-1}},
\end{aligned}$$

whence reaching

$$(2.3) \quad \kappa_{g,n} \leq \inf_{f \in C_0^1(\mathbb{R}^n)} \frac{\left( \int_{\mathbb{R}^n} |\nabla f| dv_{g,n} \right)^{\frac{n}{n-1}}}{\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} dv_{g,n}}.$$

To check the reversed inequality of (2.3), as to  $\Omega \in BDC(\mathbb{R}^n)$  and  $\epsilon > 0$  we choose the following function

$$f_\epsilon(x) = \begin{cases} 1 & , \quad x \in \Omega \\ 1 - \epsilon^{-1} \text{dist}_g(x, \partial\Omega) & , \quad x \in \mathbb{R}^n \setminus \Omega \text{ \& \; } \text{dist}_g(x, \partial\Omega) < \epsilon \\ 0 & , \quad x \in \mathbb{R}^n \setminus \Omega \text{ \& \; } \text{dist}_g(x, \partial\Omega) \geq \epsilon. \end{cases}$$

Here  $\text{dist}_g(x, \partial\Omega)$  is the distance from  $x$  to  $\partial\Omega$  with respect to the metric  $g$ . When  $\epsilon$  is small enough, we have that

$$|\nabla f_\epsilon(x)| = \begin{cases} \epsilon^{-1} & , \quad x \in \mathbb{R}^n \setminus \overline{\Omega} \text{ \& \; } \text{dist}_g(x, \partial\Omega) < \epsilon \\ 0 & , \quad \text{otherwise,} \end{cases}$$

where  $\overline{\Omega}$  is the closure of  $\Omega$ , but also that  $f_\epsilon$  tends to the characteristic function  $1_\Omega$  of  $\Omega$  as  $\epsilon \rightarrow 0$ . Hence

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0} \frac{\left( \int_{\mathbb{R}^n} |\nabla f_\epsilon| dv_{g,n} \right)^{\frac{n}{n-1}}}{\int_{\mathbb{R}^n} |f_\epsilon|^{\frac{n}{n-1}} dv_{g,n}} \\
&= \frac{\left( \lim_{\epsilon \rightarrow 0} \epsilon^{-1} v_{g,n}(\{x \in \mathbb{R}^n \setminus \Omega : \text{dist}_g(x, \partial\Omega) < \epsilon\}) \right)^{\frac{n-1}{n}}}{\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} |f_\epsilon|^{\frac{n}{n-1}} dv_{g,n}} \\
&= \frac{(s_{g,n}(\partial\Omega))^{\frac{n}{n-1}}}{v_{g,n}(\Omega)}
\end{aligned}$$

and consequently,

$$(2.4) \quad \inf_{f \in C_0^1(\mathbb{R}^n)} \frac{\left( \int_{\mathbb{R}^n} |\nabla f| dv_{g,n} \right)^{\frac{n}{n-1}}}{\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} dv_{g,n}} \leq \kappa_{g,n}.$$

Evidently, (2.3) and (2.4) imply (2.2).

*Remark 2.5.* (i) From [4, Theorem 1.3] and its odd-dimensional analog (cf. [27]) it follows that there exists a dimensional constant  $C_n \geq 1$  such that every Euclidean manifold  $(\mathbb{R}^n, g)$  with  $n \geq 3$  is  $C_n$ -biLipschitz equivalent to the background manifold  $(\mathbb{R}^n, g_0)$  – in other words –  $e^{nu}$  is comparable to the Jacobian determinant of a quasiconformal mapping from  $\mathbb{R}^n$  to itself (this guarantees that  $e^{nu}$  is a strong  $A^\infty$ -weight), and hence (1.7) holds, as long as  $u \in C^\infty(\mathbb{R}^n)$  satisfies (1.6) and

$$(2.5) \quad \int_{\mathbb{R}^n} |(-\Delta)^{n/2} u| d\mathcal{H}^n < \frac{n2^{n-1}\Gamma(n/2)\pi^{n/2}}{2^{7+4n}e^{4n(n-1)}3^{2n}}.$$

Noticing the strict inclusion  $H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ , we can immediately read off that the requirements (1.5) and (1.6) are a sufficient but not necessary condition for (1.7) to be true.

(ii) Under either the hypotheses of Theorem 1.1 or the conditions (1.6) and (2.5), we can apply [8, Theorem] to establish the following inequality concerning the best Sobolev constant for the conformal metric  $g = e^{2u}g_0$ :

$$0 < \inf_{f \in C_0^1(\mathbb{R}^n)} \frac{\left( \int_{\mathbb{R}^n} |\nabla f|^p dv_{g,n} \right)^{\frac{1}{p}}}{\left( \int_{\mathbb{R}^n} |f|^{\frac{pn}{n-p}} dv_{g,n} \right)^{\frac{n-p}{pn}}} < \infty \quad \text{where } 1 < p < n.$$

### 3. PROOF OF THEOREM 1.2

The forthcoming isoperimetric deficit formula (attached to the Chern-Gauss-Bonnet integral inequality for  $g = e^{2u}g_0$ ) is taken from the main theorems in [27] and [35].

**Lemma 3.1.** *Let  $u \in C^\infty(\mathbb{R}^n)$ . If  $g = e^{2u}g_0$  is complete conformal metric on  $\mathbb{R}^n$ ,  $n \geq 3$ , but also satisfies (1.8), then*

$$1 - \frac{\int_{\mathbb{R}^n} Q_{g,n} dv_g}{2^{n-1}\Gamma(n/2)\pi^{n/2}} = \lim_{r \rightarrow \infty} \frac{(s_g(\partial B_r(0)))^{n/(n-1)}}{(n\omega_n^{1/n})^{n/(n-1)}v_g(B_r(0))}.$$

**Proof Theorem 1.2.** This follows from Lemma 3.1, Theorem 1.1, the estimate

$$\int_{\mathbb{R}^n} |Q_{g,n}| dv_{g,n} = \|(-\Delta)^{n/2} u\|_{L^1} \leq \|(-\Delta)^{n/2} u\|_{H^1},$$

the vanishing integral condition

$$\int_{\mathbb{R}^n} (-\Delta)^{n/2} u d\mathcal{H}^n = 0 \quad \text{for } (-\Delta)^{n/2} u \in H^1(\mathbb{R}^n),$$

and the evident inequality

$$\inf_{\Omega \in BDC(\mathbb{R}^n)} \frac{(s_{g,n}(\partial\Omega))^{\frac{n}{n-1}}}{v_{g,n}(\Omega)} \leq \lim_{r \rightarrow \infty} \frac{(s_{g,n}(\partial B_r(0)))^{\frac{n}{n-1}}}{v_{g,n}(B_r(0))}.$$

Next, we handle the equality case of (1.10). If  $g = g_0$ , then  $u = 0$  which derives

$$\kappa_{g,n} = \kappa_{g_0,n} = (n\omega_n^{\frac{1}{n}})^{\frac{n}{n-1}}.$$

Conversely, suppose  $\kappa_{g,n} = (n\omega_n^{\frac{1}{n}})^{\frac{n}{n-1}}$ . Then

$$\inf_{\Omega \in BDC(\mathbb{R}^n)} \frac{(s_{g,n}(\partial\Omega))^{\frac{n}{n-1}}}{v_{g,n}(\Omega)} = (n\omega_n^{\frac{1}{n}})^{\frac{n}{n-1}}$$

Now from the formula

$$\frac{d v_{g,n}(B_r(x))}{dr} = s_{g,n}(\partial B_r(x)) \quad \text{for } x \in \mathbb{R}^n \quad \text{and } r > 0$$

it follows that

$$(n\omega_n^{\frac{1}{n}})^{\frac{n}{n-1}} \leq \frac{(s_{g,n}(\partial B_r(x)))^{\frac{n}{n-1}}}{v_{g,n}(B_r(x))} = \frac{\left(\frac{d v_{g,n}(B_r(x))}{dr}\right)^{\frac{n}{n-1}}}{v_{g,n}(B_r(x))},$$

namely,

$$n\omega_n^{\frac{1}{n}} \leq \left(v_{g,n}(B_r(x))\right)^{\frac{1}{n}-1} \frac{d v_{g,n}(B_r(x))}{dr}.$$

An integration acting on this last inequality gives

$$(3.1) \quad \omega_n r^n \leq v_{g,n}(B_r(x)).$$

On the other hand, the geometric interpretation of the scalar curvature reveals (cf. [13, 3.98 Theorem])

$$\frac{v_{g,n}(B_r(x))}{\omega_n r^n} = 1 - \frac{S_{g,n}(x)}{6(n+2)} r^2 + o(r^2) \quad \text{as } r \rightarrow 0.$$

Since  $S_{g,n}(x) \geq 0$  for  $x \in \mathbb{R}^n$ , we conclude

$$(3.2) \quad \lim_{r \rightarrow 0} \frac{v_{g,n}(B_r(x))}{\omega_n r^n} \leq 1.$$

Using the previous estimates (3.1)-(3.2) and the fundamental theorem of Lebesgue (cf. [30, pp.4-5]), we find

$$e^{nu(x)} = \lim_{r \rightarrow 0} (\omega_n r^n)^{-1} \int_{B_r(x)} e^{nu} d\mathcal{H}^n = \lim_{r \rightarrow 0} \frac{v_{g,n}(B_r(x))}{\omega_n r^n} = 1 \quad \text{for } x \in \mathbb{R}^n,$$

whence getting  $u = 0$  and so  $g = g_0$ .

*Remark 3.2.* (i) Under the equality result of Theorem 1.2, the proof of [15, Proposition 8.2], along with the extremal function

$$f(x) = (1 + |x|^{\frac{p}{p-1}})^{\frac{p-n}{p}} \quad \text{for } x \in \mathbb{R}^n,$$

yields that for any  $p \in (1, n)$  the well-known best Sobolev constant

$$\inf_{f \in C_0^1(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} |\nabla f|^p dv_{g_0,n}\right)^{\frac{1}{p}}}{\left(\int_{\mathbb{R}^n} |f|^{\frac{pn}{n-p}} dv_{g_0,n}\right)^{\frac{n-p}{pn}}}$$

is equal to

$$\left(\frac{n^{\frac{1}{p-1}}(n-p)}{p-1}\right)^{1-\frac{1}{p}} \left(\frac{\omega_n \Gamma(\frac{n}{p}) \Gamma(1+n-\frac{n}{p})}{\Gamma(n)}\right)^{\frac{1}{n}}.$$

It seems natural to conjecture that for any complete conformal metric  $g = e^{2u} g_0$  satisfying (1.9), the inequality

$$\inf_{f \in C_0^1(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} |\nabla f|^p dv_{g,n}\right)^{\frac{1}{p}}}{\left(\int_{\mathbb{R}^n} |f|^{\frac{pn}{n-p}} dv_{g,n}\right)^{\frac{n-p}{pn}}} \leq \left(\frac{n^{\frac{1}{p-1}}(n-p)}{p-1}\right)^{1-\frac{1}{p}} \left(\frac{\omega_n \Gamma(\frac{n}{p}) \Gamma(1+n-\frac{n}{p})}{\Gamma(n)}\right)^{\frac{1}{n}}$$

holds and the last equality happens when and only when  $g = g_0$ . Obviously, the last infimum is positive under the above-pointed suppositions.

(ii) Maybe it is appropriate to recall the so-called “non-compact Yamabe problem”, which states: *On a smooth, complete, non-compact  $3 \leq n$ -dimensional Riemannian manifold  $(M, g)$ , does there exist a complete conformal metric of constant scalar curvature?* Although this problem was answered negatively through Z. Jin’s counterexample in [19], it would still be of independent interest to find a criterion for the 1-scalar curvature equation

$$(3.3) \quad S_{g,n} = -2(n-1)e^{-2u} \left( \Delta u + \frac{n-2}{2} |\nabla u|^2 \right) = 1$$

to be solvable in a suitable function space. From Theorem 1.2 it is seen that if this equation has a solution  $u$  belonging to  $C^\infty(\mathbb{R}^n)$  and obeying  $(-\Delta)^{n/2}u \in H^1(\mathbb{R}^n)$  then (1.10) holds. A follow-up question arises: *Is (1.10) a sufficient condition for the existence of a solution to (3.3)?*

#### 4. PROOF OF THEOREM 1.3

To prove Theorem 1.3 let us review the so-called  $C^1$  Sard type theorem (cf. [29, Theorem 10.4])

**Lemma 4.1.** *Given a bounded domain  $\Omega \subset \mathbb{R}^n$  with  $n \geq 2$  let  $f$  be a real-valued  $C^1$  function on  $\Omega$  with*

$$\sup_{x \in \Omega} (|f(x)| + |\nabla f(x)|) < \infty.$$

*Then*

$$f^{-1}(t) = (f^{-1}(t) \setminus \{x \in \Omega : \nabla f(x) = 0\}) \cup (f^{-1}(t) \cap \{x \in \Omega : \nabla f(x) = 0\})$$

*holds for almost all  $t \in f(\Omega)$ , where  $f^{-1}(t) \setminus \{x \in \Omega : \nabla f(x) = 0\}$  is an  $(n-1)$ -dimensional  $C^1$ -submanifold with*

$$\mathcal{H}^{n-1}(f^{-1}(t) \cap \{x \in \Omega : \nabla f(x) = 0\}) = 0 \quad \text{and} \quad \mathcal{H}^{n-1}(f^{-1}(t)) < \infty.$$

*Consequently, if  $S_f$  consists of the above  $t$ ’s then  $\mathcal{H}^1(f(\Omega) \setminus S_f) = 0$ .*

With the help of Lemma 4.1 and the asymptotic behavior of the Green’s function of  $\Omega \in BRD(\mathbb{R}^n)$  below:

$$G_{\Omega,n}(x, y) = -(n\omega_n)^{\frac{1}{1-n}} \log |x - y| + O(1) \quad \text{as } x \rightarrow y \text{ in } \mathbb{R}^n,$$

W. Wang discovered an integral formula for the  $n$ -Green function (cf. [32, Lemma 4.1]) as follows.

**Lemma 4.2.** *Let  $y \in \Omega \in BRD(\mathbb{R}^n)$  with  $n \geq 2$ . Then*

$$\int_{\{x \in \Omega : G_{\Omega,n}(x, y) = t\}} |\nabla G_{\Omega,n}(\cdot, y)|^{n-1} d\mathcal{H}^{n-1}(\cdot) = 1$$

*holds for each  $t \in S_{G_{\Omega,n}(\cdot, y)}$ .*

**Proof of Theorem 1.3.** (i) For  $t \geq 0$  and  $y \in \Omega \in BRD(\mathbb{R}^n)$  set

$$\Omega(t, y; G) = \{x \in \Omega : G_{\Omega,n}(x, y) \geq t\}.$$

Then  $G_{\Omega,n}(\cdot, y)$  is of  $C^1$  class on  $\Omega \setminus \{y\}$ , and hence for  $t \in S_{G_{\Omega,n}(\cdot, y)}$  we have

$$\partial\Omega(t, y; G) = \{x \in \Omega : G_{\Omega,n}(x, y) = t\},$$

which is the pre-image of  $t$  under  $G_{\Omega,n}(\cdot, y)$ . From now on, we will assume

$$F(t, y) = v_{g,n}(\Omega(t, y; G)) = \int_{\Omega(t, y; G)} e^{nu} d\mathcal{H}^n.$$

On  $S_{G_{\Omega,n}}$  this function decreases – in fact  $F(t, y)$  enjoys the differential equation (cf. [1, p.53, Lemma 2.5])

$$(4.1) \quad -\frac{dF(t, y)}{dt} = \int_{\partial\Omega(t, y; G)} \frac{e^{nu(x)}}{|\nabla G_{\Omega}(x, y)|} d\mathcal{H}^{n-1}(x) \quad \text{for } t \in S_{G_{\Omega,n}}.$$

Applying Hölder's inequality, Lemma 4.2 and Theorem 1.1, we further derive from (4.1) that for  $t \in S_{G_{\Omega,n}}$ ,

$$\begin{aligned} & \left( -\frac{dF(t, y)}{dt} \right)^{\frac{n-1}{n}} \\ &= \left( \int_{\partial\Omega(t, y; G)} \frac{e^{nu(x)}}{|\nabla G_{\Omega}(x, y)|} d\mathcal{H}^{n-1}(x) \right)^{\frac{n-1}{n}} \left( \int_{\partial\Omega(t, y; G)} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla G_{\Omega}(x, y)|^{1-n}} \right)^{\frac{1}{n}} \\ &\geq \int_{\partial\Omega(t, y; G)} e^{(n-1)u(x)} d\mathcal{H}^{n-1}(x) \\ &\geq \kappa_{g,n}^{\frac{n-1}{n}} \left( v_{g,n}(\Omega(t, y; G)) \right)^{\frac{n-1}{n}} \\ &= \kappa_{g,n}^{\frac{n-1}{n}} (F(t, y))^{\frac{n-1}{n}}. \end{aligned}$$

The above inequalities yield

$$\frac{d}{dt} \left( e^{\kappa_{g,n} t} F(t, y) \right) = e^{\kappa_{g,n} t} \left( \kappa_{g,n} F(t, y) + \frac{dF(t, y)}{dt} \right) \leq 0.$$

In other words,  $e^{\kappa_{g,n} t} F(t, y)$  decreases with  $t \in S_{G_{\Omega,n}}$ .

Because Lemma 4.1 illustrates

$$\mathcal{H}^1(\{t = G_{\Omega,n}(x, y) \in (0, \infty] : x \in \Omega\} \setminus S_{G_{\Omega,n}}) = 0,$$

we can treat  $F(\cdot, y)$  as a continuous and decreasing function on  $[0, \infty)$  but also  $e^{\kappa_{g,n} t} F(t, y)$  as a decreasing function with  $t \in [0, \infty)$ . Note that if  $p > 0$  and

$$F_p(t, y) = \int_{\Omega(t, y; G)} (G_{\Omega,n}(x, y))^p e^{nu(x)} d\mathcal{H}^n(x),$$

then

$$F_p(0, y) = \int_{\Omega} (G_{\Omega,n}(x, y))^p e^{nu(x)} d\mathcal{H}^n(x)$$

and hence, using the layer cake representation and integrating by part, we deduce

$$F_p(t, y) = - \int_t^\infty r^p dF(r, y).$$

So, without loss of generality we may assume  $F_q(0, y) < \infty$  for  $0 \leq q < p < \infty$  – otherwise there is nothing to argue. Since  $d(e^{\kappa_{g,n} t} F(t, y))/dt \leq 0$ , we conclude (via an integration by part) that

$$F_q(t, y) \leq \kappa_{g,n} e^{\kappa_{g,n} t} \int_t^\infty r^q e^{-\kappa_{g,n} r} dr$$

and consequently,

$$\frac{d}{dt} \log F_q(t, y) \leq \frac{d}{dt} \log \int_t^\infty r^q e^{-\kappa_{g,n} r} dr.$$

Integrating this last differential inequality from 0 to  $t$ , we get

$$\frac{F_q(t, y)}{F_q(0, y)} \leq \frac{\kappa_{g,n}^{1+q}}{\Gamma(1+q)} \int_t^\infty r^q e^{-\kappa_{g,n} r} dr.$$

This estimate produces

$$\begin{aligned} & F_p(0, y) \\ &= - \int_0^\infty t^{p-q} t^q dF(t, y) \\ &= (p-q) \int_0^\infty t^{p-q-1} F_q(t, y) dt \\ &\leq \frac{(p-q) \kappa_{g,n}^{q+1} F_q(0, y)}{\Gamma(q+1)} \int_0^\infty t^{p-q-1} \left( \int_t^\infty r^q e^{-\kappa_{g,n} r} dr \right) dt \\ &= \kappa_{g,n}^{q-p} \left( \frac{\Gamma(p+1)}{\Gamma(q+1)} \right) F_q(0, y), \end{aligned}$$

which in turn verifies (1.11).

Furthermore,  $g = g_0$  implies  $u = 0$  and  $\kappa_{g,n} = (n\omega_n^{\frac{1}{n}})^{\frac{n}{n-1}}$ . Now that

$$(4.2) \quad G_{B_1(0),n}(0, y) = -(n\omega_n)^{\frac{1}{1-n}} \log |y| \quad \text{for } y \in B_1(0),$$

$g = g_0$  yields also

$$\begin{aligned} & \frac{\kappa_{g,n}^{-q} \Gamma(q+1)}{\kappa_{g,n}^{-p} \Gamma(p+1)} \\ &\leq \inf_{x \in \Omega \in BRD(\mathbb{R}^n)} \frac{\int_\Omega (G_{\Omega,n}(x, y))^q dv_{g,n}(y)}{\int_\Omega (G_{\Omega,n}(x, y))^p dv_{g,n}(y)} \\ &\leq \frac{\int_{B_1(0)} (G_{B_1(0),n}(0, y))^q d\mathcal{H}^n(y)}{\int_{B_1(0)} (G_{B_1(0),n}(0, y))^p d\mathcal{H}^n(y)} \\ &= \frac{(n\omega_n)^{1-\frac{q}{n-1}} \int_0^1 \left( \log \frac{1}{r} \right)^q r^{n-1} dr}{(n\omega_n)^{1-\frac{p}{n-1}} \int_0^1 \left( \log \frac{1}{r} \right)^p r^{n-1} dr} \\ &= \frac{(n\omega_n^{\frac{1}{n}})^{-\frac{qn}{n-1}} \Gamma(q+1)}{(n\omega_n^{\frac{1}{n}})^{-\frac{pn}{n-1}} \Gamma(p+1)}. \end{aligned}$$

Thus, the equality in (1.11) occurs.

(ii) From Theorem 1.1 and the case  $0 = q < p < \infty$  of (i) it follows that  $\kappa_{g,n} > 0$  and for any  $x \in \Omega \in BRD(\mathbb{R}^n)$ ,

$$\frac{\kappa_{g,n}^p}{\Gamma(p+1)} \leq \frac{v_{g,n}(\Omega)}{\int_\Omega (G_{\Omega,n}(x, \cdot))^p dv_{g,n}(\cdot)} \leq \frac{\kappa_{g,n}^{-1} (s_{g,n}(\partial\Omega))^{\frac{n}{n-1}}}{\int_\Omega (G_{\Omega,n}(x, \cdot))^p dv_{g,n}(\cdot)}.$$

This derives (1.12). When  $g = g_0$ , as done in the last part of the foregoing (i) a calculation with (4.2) yields

$$\frac{\kappa_{g_0,n}^{p+1}}{\Gamma(p+1)} = \frac{(n\omega_n^{\frac{1}{n}})^{\frac{n(p+1)}{n-1}}}{\Gamma(p+1)} = \frac{(s_{g_0,n}(\partial B_1(0)))^{\frac{n}{n-1}}}{\int_{B_1(0)} (G_{B_1(0),n}(0, \cdot))^p dv_{g_0,n}(\cdot)},$$

whence reaching the equality of (1.12).

*Remark 4.3.* (i) We have not been able to prove whether or not the equality of either (1.11) or (1.12) implies  $g = g_0$ . Nevertheless we strongly conjecture that it has an affirmative answer.

(ii) When  $w$  is the Jacobian determinant  $J_f$  of a quasiconformal map  $f$  from  $\mathbb{R}^n$  to itself,  $w$  is a strong  $A_\infty$ -weight and so by Lemma 2.2,

$$\kappa_w = \inf_{\Omega \in BDC(\mathbb{R}^n)} \frac{\left( \int_{\partial\Omega} w^{\frac{n-1}{n}} d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}}}{\int_{\Omega} w d\mathcal{H}^n} > 0.$$

A careful look at the proof of Theorem 1.3 indicates that this theorem is still true with  $\kappa_w$  replacing  $\kappa_{g,n}$ . In particular,

$$\int_{\Omega} (G_{\Omega,n}(\cdot, y))^p J_f(\cdot) d\mathcal{H}^n(\cdot) \leq \frac{\Gamma(p+1)}{\kappa_w^p} \int_{\Omega} J_f d\mathcal{H}^n,$$

where  $y \in \Omega \in BDC(\mathbb{R}^n)$  and  $0 \leq p < \infty$ . This observation suggests a future study of the quasiregular  $Q_p$ -space  $QRQ_p(\Omega; \mathbb{R}^n)$  which comprises all quasiregular mappings  $f : \Omega \rightarrow \mathbb{R}^n$  with

$$\sup_{y \in \Omega} \int_{\Omega} (G_{\Omega,n}(x, y))^p |f'(x)|^n d\mathcal{H}^n(x) \approx \sup_{y \in \Omega} \int_{\Omega} (G_{\Omega,n}(x, y))^p J_f(x) d\mathcal{H}^n(x) < \infty.$$

Here  $f'(x)$  means the formal derivative of  $f$  at  $x$ , that is, the matrix  $[\partial f_j(x)/\partial x_k]_{n \times n}$  of the partial derivatives  $\partial f_j(x)/\partial x_k$ ,  $j, k = 1, \dots, n$ , of the coordinate functions  $f_1, \dots, f_n$  of  $f$ . Moreover,  $|f'(x)| = \max_{h \in \partial B_1(0)} |f'(x)h|$ . And, a continuous mapping  $f : \Omega \rightarrow \mathbb{R}^n$  is called quasiregular provided that its coordinate functions  $f_1, \dots, f_n$  lie in the local homogeneous  $n$ -Sobolev space  $\dot{W}_{loc}^{1,n}(\Omega)$ , i.e.,

$$\int_O |\nabla f_j|^n d\mathcal{H}^n < \infty, \quad j = 1, \dots, n$$

for each open set  $O$  compactly contained in  $\Omega$ , and that there exists a constant  $\mathcal{K} \geq 1$  such that

$$(4.3) \quad J_f(x) \leq |f'(x)|^n \leq \mathcal{K} J_f(x)$$

is valid for almost all  $x \in \Omega$ . Especially, the quasiregular homeomorphism is said to be a quasiconformal mapping. When  $n = 2$  and  $\mathcal{K} = 1$  in (4.3) the concept of quasiregular/quasiconformal returns to the concept of holomorphic/conformal. See also: [16] for more information on the quasiregular mappings, [33]-[34] for an overview of the recent research results on the holomorphic and geometric  $Q_p$ -spaces on the unit disk of  $\mathbb{R}^2$ , and [22] for an investigation of the  $Q_p$ -type function space over  $B_1(0)$  introduced by a kind of invariance under Möbius transformations.

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